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On Borromean links

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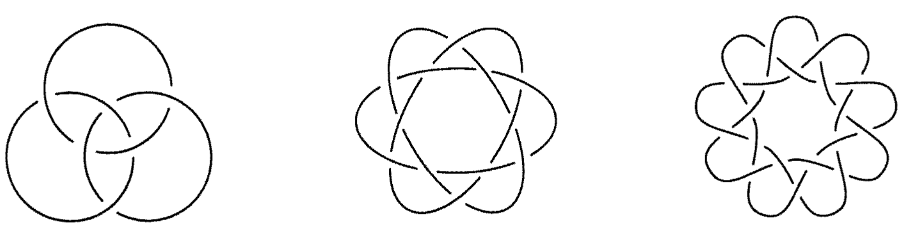
n-Borromean links are nontrivial links in which n rings, $n \geq 3$, are combined in such a way that any two component rings form a trivial link. The symmetry of links with $n = 3$ is discussed, and it is shown that such links form a variety of series whose members are different isotopy types. Examples are adduced of 3-Borromean links that are topologically chiral. Novel constructions are described of n-Borromean links with and without at least one nontrivial sublink.

1. Introduction

The Borromean (or Ballantine) link (fig. 1(a)) is among the most fascinating of topological constructions: three mutually disjoint simple closed curves form a link, yet no two curves are linked. Thus, if any one curve is cut, the other two are free to separate. An "elementary proof" has recently been published [1]. Though this classic link is of ancient provenance, to our knowledge its curious characteristic was first explicitly noted by Tait [2], who also showed that this link is the first member in a series of links that "are formed by three unknotted closed curves, no two of which are linked together", and in which "the number of crossings is a multiple



(a) (b) (c)

Fig. 1. Reduced diagrams of Borromean links in Tait's series. (a) The Ballantine link, with 6 crossings (fig. 15 in Plate XV of [2], denoted 6^3_2 in [3a]). (b) The link with 12 crossings (fig. 18 in Plate XV of [2]). (c) The link with 18 crossings.

of six". The members of Tait's series are not mutually interconvertible by continuous deformation; that is, they are different isotopy types. Figures 1(a)–(c) depict reduced diagrams of the first three members of the series.

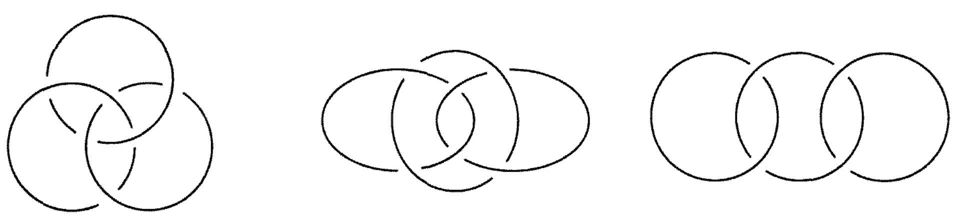
The Ballantine link (6_2^3 in Rolfsen's notation [3a]) is one of three 3-component prime links with six crossings, but the other two (6_3^3 in fig. 2(a) and 6_1^3 in fig. 2(b)) do not share the peculiar features described above. Given its unique construction, it comes as no surprise that realization of the Ballantine link in molecular form is considered a synthetic goal well worth achieving. As Martin Gardner put it, "Who can guess what outlandish properties a carbon compound might have . . . if its molecules were joined into triplets, each triplet interlocked like a set of Borromean rings?" [4]. More than 30 years have passed since Wasserman expressed the view that molecular Borromean links "require a minimum string of 30 carbons" in each of the three rings [5], and since van Gulick discussed the 3-braid approach to the synthesis of such a link [6], yet the synthetic goal remains elusive. Assuredly, "the synthesis of the Borromean ring system is still a challenging problem to chemists" [7], and "certainly one day molecular Borromean rings will be created by a directed approach" [8], but at present that day still lies in the future. The closest that chemists have come so far is the synthesis of [3]-catenanes, molecules whose structure corresponds to the 4-crossing product link in fig. 2(c) [9,10].

The present paper was motivated by the continuing interest of chemists and mathematicians in this remarkable construction.

2. Symmetry of 3-Borromean links

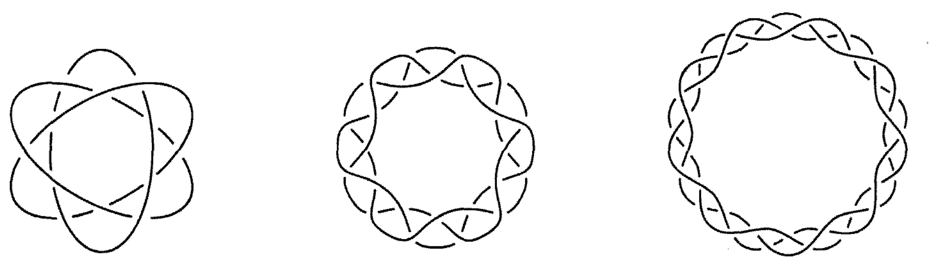
We define a *n-Borromean link* as a nontrivial link in which n rings, $n \geq 3$, are combined in such a way that any two component rings form a trivial link. By "ring" we mean an unknotted closed (smooth or polygonal) curve. According to this definition, all the members of Tait's series, including the Ballantine link, are 3-Borromean links.

It is easily shown that Tait's links can all assume rigidly achiral presentations with S_m symmetry, where m is the number of crossings in the reduced diagram.



(a) (b) (c)

Fig. 2. (a) 6_3^3 (fig. 26 in Plate XVI of [2]) and (b) 6_1^3, the other two 3-component prime links with six crossings [3a]. (c) The 3-component product link $2_1^2\#2_1^2$.

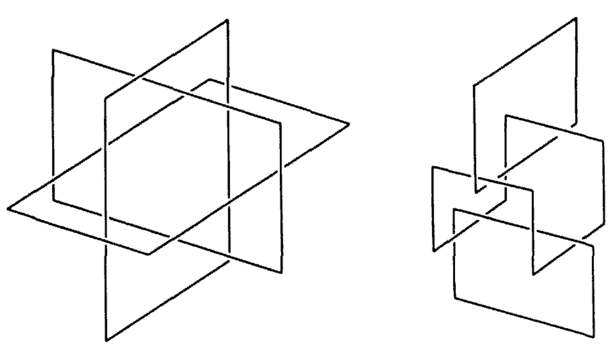


(a) (b) (c)

Fig. 3. Reduced diagrams of rigidly achiral (S_m symmetry) presentations of the links in fig. 1. (a) $m = 6$, (b) $m = 12$, (c) $m = 18$.

Thus all the members of Tait's series are topologically achiral (amphicheiral [11]). Amphicheirality is rare among reported links: all but three (2_1^2, 6_2^2, 8_8^2) of the 91 prime links with two component rings and with 9 or fewer crossings listed by Rolfsen [3a], and all but three (6_2^3, 8_4^3, 8_6^3) of the 35 3-component prime links with 9 or fewer crossings [3a], are topologically chiral.

Figure 3 depicts S_m diagrams of the first three members of Tait's series. The Ballantine link (figs. 1(a) and 3(a)) is the only member of the series that can additionally assume a rigidly achiral presentation with T_h symmetry, in which the three rings lie in three mutually perpendicular planes. This geometry had previously been recognized by van Gulick [6]. Shortly thereafter, Tauber [12] pointed out that the achirality of the T_h presentation persists even when all three rings are oriented; this is in contrast to the achiral 2-component link 2_1^2, which famously acquires topological chirality upon orientation of both rings. It is of interest to note that three golden rectangles in mutually perpendicular planes (whose twelve vertices are the twelve vertices of the regular icosahedron) [13] also constitute a T_h-symmetric Ballantine link (fig. 4(a))! In addition to presentations with T_h subsymmetries (including S_6), the Ballantine link can also assume D_{2d} symmetry



(a) (b)

Fig. 4. Additional rigidly achiral presentations of the Ballantine link. (a) T_h symmetry: golden rectangles in mutually perpendicular planes [13]. (b) D_{2d} symmetry.

(fig. 4(b)). It remains to be noted that Ballantine links composed of metric *circles* are impossible [14], and that Ballantine links composed of *squares* have inspired sculptures [15] whose highest attainable symmetry is S_6.

While all the members of Tait's series are 3-Borromean links, the converse is certainly not true. For example, figs. 5(a)–(c) display the second, third, and fourth members in a series of 3-Borromean links whose first member is the Ballantine link; while topologically achiral like Tait's links, however, the members of this new series have $4m + 2$, $m = 1, 2, \ldots$, instead of $6m$ crossings. As another kind of example, figs. 5(d)–(f) display the second, fourth, and sixth members in a series of 3-Borromean links with $m + 12$ crossings, $m = 0, 1, 2, \ldots$, whose first member is shown in fig. 1(b). By analogy with alternating knots [11], the alternating links with an odd number of crossings are conjectured to be topologically chiral, unlike the links in Tait's series. From these examples it becomes evident that there must exist numerous series of n-Borromean links, and that members of these series can be topologically chiral as well as achiral, as illustrated in fig. 5.

3. Generalized n-Borromean links

In principle we can distinguish between two kinds of n-Borromean links: those in which every sublink is trivial, and those with at least one nontrivial sublink. The

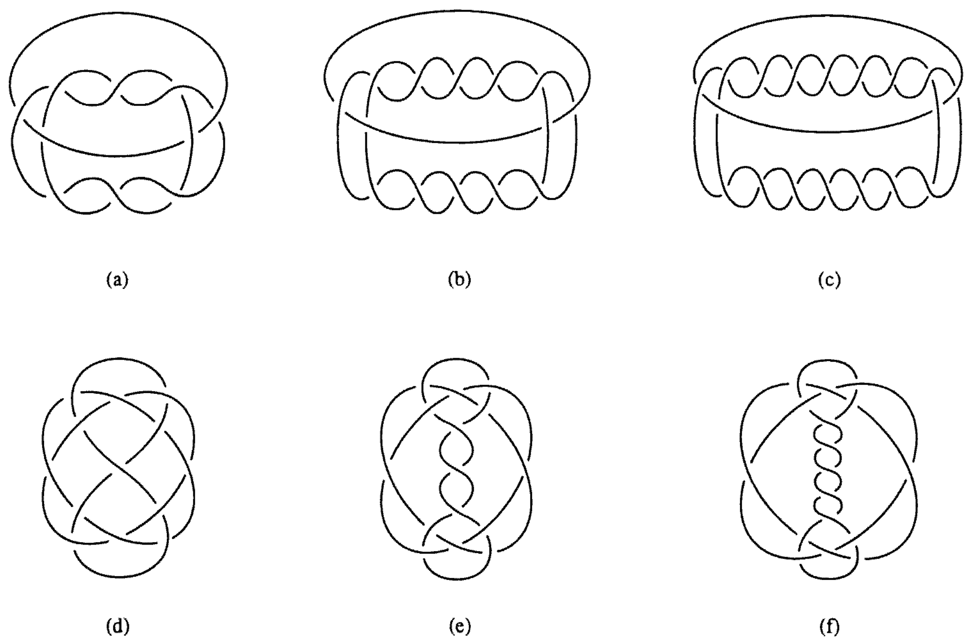


Fig. 5. Reduced diagrams of 3-Borromean links. Members of the $4m + 2$ series are shown with (a) 10, (b) 14, and (c) 18 crossings. All the members of this series have attainable rigid C_s symmetry. Members of the $m + 12$ series are shown with (d) 13, (e) 15, and (f) 17 crossings. All the members of this series with m = odd are conjectured to be topologically chiral.

first kind are called [3b] *Brunnian links* and are said to have the *Brunnian property*, in honor of an early contribution by Hermann Brunn to knot theory [16,17].

3.1. BORROMEAN LINKS WITH THE BRUNNIAN PROPERTY

Construction of *n-Brunnian links* (Brunnian links composed of n rings) was begun by Tait [2], as described above, and additional methods for constructing such links were described by Brunn [16]. Here we suggest a simple construction that yields n-Brunnian links with attainable rigid D_{2d} symmetry.

Figure 6(a) shows a generalized n-Brunnian link with $n \geq 3$; for $n = 3$ this corresponds to fig. 4(b). The two adjacent rings at one end, 1 and 2, are topologically equivalent, and so are the two adjacent rings at the other end, n and $n-1$. Under D_{2d} symmetry, rings 1 and n, 2 and $n-1$, etc., are pairwise symmetry-equivalent. Therefore rings 1, 2, $n-1$, and n are pairwise topologically equivalent. It follows that all the rings in 3- and 4-Brunnian links are pairwise topologically equivalent, and that four of the rings in n-Brunnian links with $n > 4$ are pairwise topologically equivalent. Finally, as shown in fig. 6(b), when rings 2 and $n-1$ of the construction

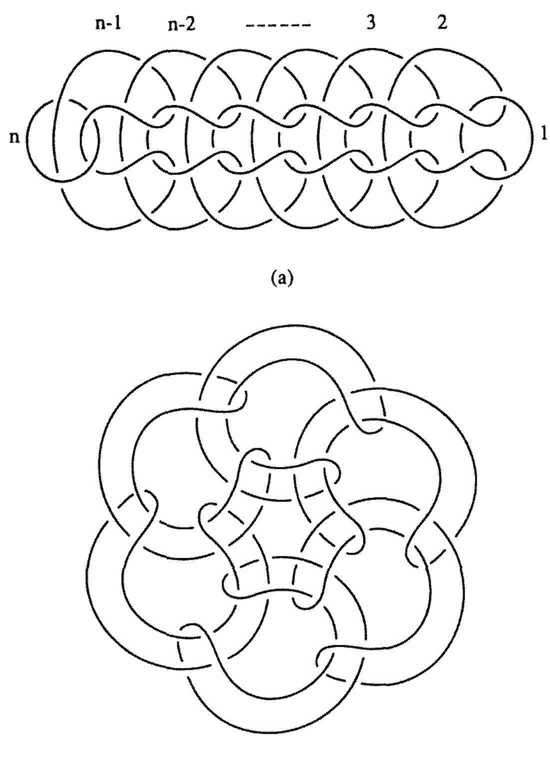


Fig. 6. Construction of n-Brunnian links with (a) attainable rigid D_{2d} symmetry; (b) attainable rigid C_{nh} symmetry (C_{6h} in this example).

in fig. 6(a) are linked without end rings 1 and n (fig. 11 of [16]), the result is a cyclic Brunnian link with attainable rigid C_{nh} symmetry.

The algorithm implied in fig. 6(a) transparently leads to the generation of a series of n-Brunnian links whose first member ($n = 3$) is the Ballantine link. This algorithm is incapable, however, of generating any of the higher members of the series with $6m$ or with $4m + 2$ crossings.

3.2. BORROMEAN LINKS WITHOUT THE BRUNNIAN PROPERTY

The distinction between Borromean links with and without the Brunnian property can be quantitatively expressed in terms of Brunn's Zerschneidungszahlen (cutting numbers) [16]. Consider a set of cuts that are successively applied to the rings of a link, and in which each cut separates the cut ring from the link. The *minimum cutting number* μ is the smallest number of cuts that suffice to unlink all the remaining (uncut) links, while the *maximum cutting number* M is the largest number of cuts that can be applied to unlink all the remaining (uncut) rings provided that none of the cuts are applied to a freed (unlinked) ring. For example, $\mu = 1$ and $M = 2$ for the 3-component product link in fig. 2(c), and $\mu = M = 1$ for all 2-component links. Thus the distinction between Borromean links with and without the Brunnian property is that $\mu = M = 1$ for the former and $M > 1$ for the latter.

We recognize two methods for constructing n-Borromean links with at least one nontrivial sublink. In the first method, which involves duplication of one or more rings, the duplicate rings are interchangeable by continuous deformation, whereas in the second method they are not. The first method is illustrated by the 4-Borromean link in fig. 7(b), which is constructed by duplicating one of the rings in the Ballantine link of fig. 7(a). The second method, which is somewhat less obvious, and which at this stage of development is still highly empirical, is illustrated by the 12-crossing 4-Borromean link in fig. 7(c). If either one of the α rings is cut, an intact Ballantine link remains, whereas cutting either one of the two rings labeled β frees the three remaining rings. Hence $\mu = 1$ and $M = 2$ for both of the 4-Borromean links shown in this figure.

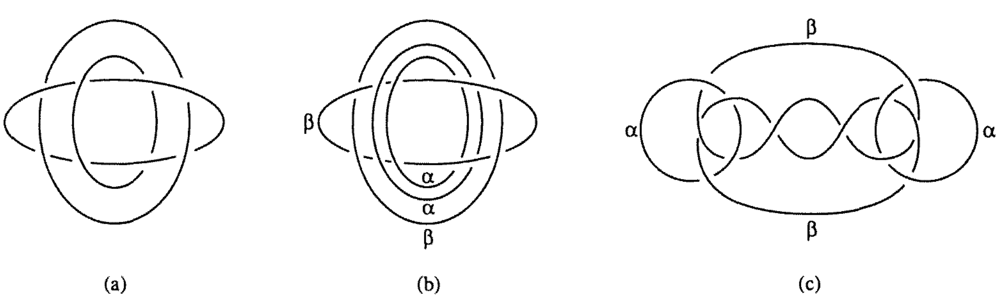


Fig. 7. (a) Ballantine link in C_i presentation. (b) 4-Borromean link, with α-rings interchangeable by continuous deformation. (c) 4-Borromean link, with α-rings not interchangeable by continuous deformation.

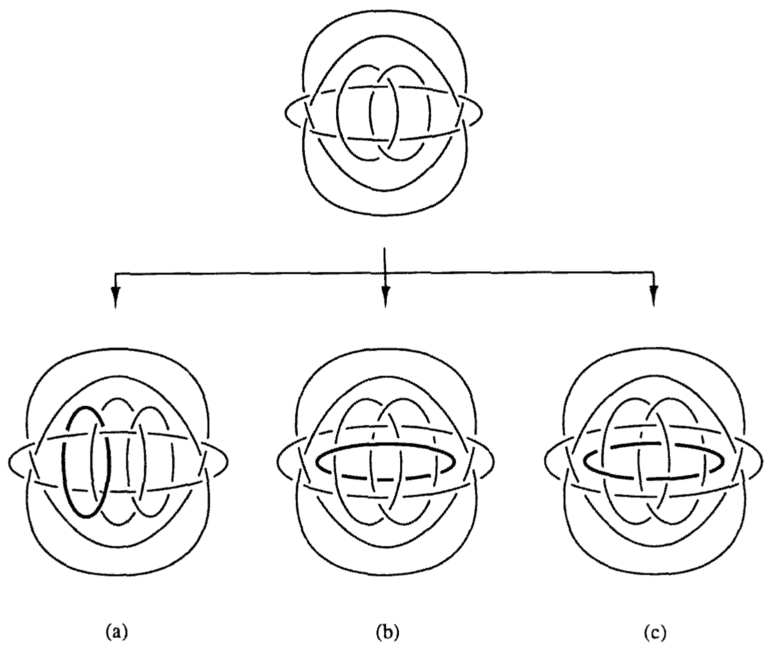


Fig. 8. Top: 5-Brunnian link in C_i presentation. Addition of one ring by the second method (see text) affords different types of 6-Borromean links without the Brunnian property. Three such links are shown (added ring emphasized) with cutting numbers (a) $\mu = 1$, $M = 2$; (b) $\mu = 1$, $M = 2$; (c) $\mu = 1$, $M = 3$.

Different modes of ring addition by the second method yield different types of links. This is illustrated in fig. 8 with the addition of one ring to a 5-Brunnian link and in fig. 9 by the addition of two rings to a 4-Brunnian link. The 6-Borromean links shown in these two figures represent five of many different isotopy types.

The unlinking pathway for a n-Borromean link with $M > 1$ depends on the order in which the rings are cut, as illustrated in fig. 10 for the link in fig. 8(c). Cutting ring 1 disconnects the five remaining rings, hence $\mu = 1$. A Ballantine link is obtained by cutting ring 3, a 4-Borromean link without the Brunnian property by cutting ring 4, and a 5-Brunnian link by cutting ring 2; cutting any one ring in the

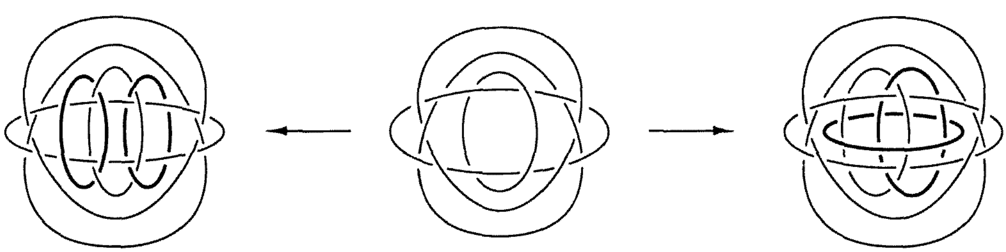


Fig. 9. 4-Brunnian link in C_i presentation. Addition of two rings by the second method (see text) affords different types of 6-Borromean links without the Brunnian property. Two such links are shown (added ring emphasized), both with cutting numbers $\mu = 1$, $M = 3$.

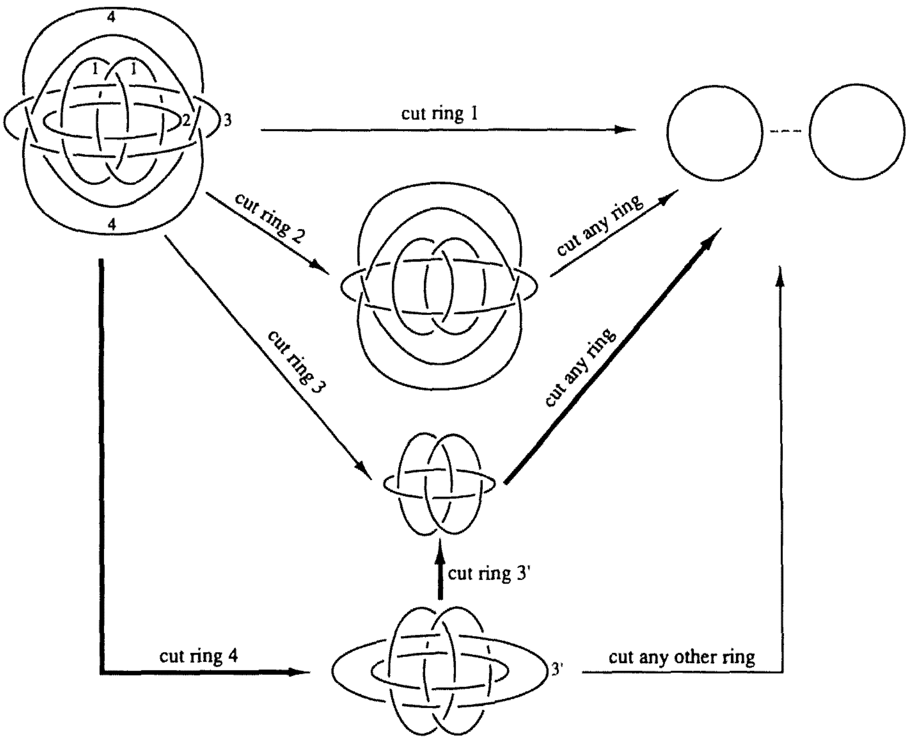


Fig. 10. Unlinking pathways for the 6-Borromean link in fig. 8(c). The minimum cutting number, $\mu = 1$, is obtained by cutting ring 1, which frees all the remaining rings. The maximum cutting number, $M = 3$, is obtained by following the path marked with heavy arrows: first cut ring 4, then ring 3′ of the resulting 4-Borromean link, and then any ring in the resulting Ballantine link.

Ballantine or 5-Brunnian links, or any ring other than the one labeled 3′ in the 4-Borromean link, disconnects all the remaining rings. The only pathway with a maximum cutting number, $M = 3$, is indicated by heavy arrows in fig. 10.

Brunn [16] had noted that, in general, minimum and maximum cutting numbers for chains and links must be determined by trial and error. This is certainly the case also for the maximum cutting number of Borromean rings constructed by the second method. Further work is required for the development of a systematic algorithm.

Acknowledgement

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